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On the non-standard representation of linear mappings from a function space*)

by

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Introduction

The study of non-standard representations of linear forms goes back to Robinson [11], where he showed that every linear form on the space of bounded sequences can be represented by a $*$-finite operator (see definition below). Luxemburg [8] generalized this result to more general spaces and proved that the second dual space of a normed linear space X can be imbedded in $^*X$. On the other hand, non-standard measure theories have been developed by Bernstein, Wattenberg, Henson, Loeb and others in [1]~[7], [10] etc.

The purpose of this work is to provide a simple method to attack above problems in a general and unified way. In particular, following results will be generalized and given a simple proof:

(1) the result of Robinson on linear forms [11],

(2) the result of Luxemburg on the second dual space [8],

(3) the result of Robinson on distributions [12],

(4) the result of Bernstein and Wattenberg on the Lebesgue measure [1] [3],

(5) the result of Henson on non-atomic probability measures [4].

We suppose that the reader is familiar with elements of non-standard analysis which can be found in [9], [12], [13] or [14].

This paper consists of three parts which are logically independent of one auother, though they are in a same frame of thought.

Part I deals with the problem of non-standard representations of linear mappings on a function space in a very general situation. Let X be a set, K a topological field, V and W topological linear spaces over K, Z the space of continuous linear mappings from V to W and E a linear space of V-valued functions on X. If h is a Z-valued finite function (i.e. with finite support) on X, a finite operator T_h: $E \longrightarrow W$ is defined by $T_h(f)=\sum_{x\in X}\langle f(x), h(x)\rangle$, where $\langle\ ,\ \rangle$ is the coupling: $V\times Z \longrightarrow W$.

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For a certain class of topological linear spaces V including Hausdorff locally convex spaces over $\mathbf{R}$ or $\mathbf{C}$, we shall prove that every linear mapping from E to W can be represented by a $*$-finite operator ${}^*T_\varphi\colon f \to {}^*\sum_{\xi\in{}^*X} \langle {}^*f(\xi), \varphi(\xi)\rangle$ (Theorem 1.5). As a Corollary, we obtain the result of Luxemburg [8]. We apply Theorem 1.5 to reproduce and generalize the result of Robinson on distributions [12] (Theorems 1.10 and 1.11).

In Part II, we shall first prove that every conic form with values in $\mathbf{R}^+\cup\{+\infty\}$ on a conic space (see Definition 2.1) of functions on X can be represented up to an infinitesimal by a $*$-finite operator with positive kernel whose support includes X (Theorem 2.3). As its consequence, if $(X, \boldsymbol{B}, m)$ is an arbitrary measure space, the integral $\int f\,dm$ for every measurable $\boldsymbol{R}$-valued function f having definite integral value in $\overline{\boldsymbol{R}}=\boldsymbol{R}\cup\{-\infty, +\infty\}$ can be represented by a $*$-finite operator of the above type (Theorem 2.5). If, in Theorem 2.3, X is a Hausdorff space without isolated point and if the functions in E are all continuous, then the $*$-kernel can be taken as the characteristic function of a $*$-finite set divided by a $*$-natural number (Theorem 2.10).

Part III deals with a non-atomic measure space $(X, \boldsymbol{B}, m)$ and proves that the integral $\int f\,dm$ for every measurable $\boldsymbol{R}$-valued function having definite integral value in $\overline{\boldsymbol{R}}$ can be represented (up to an infinitesimal) in the form $1/\rho\cdot{}^*\sum_{\xi\in\Gamma}{}^*f(\xi)$ where Γ is a $*$-finite subset of ${}^*X$ including X and ρ is a $*$-natural number (Theorem 3.3). If in particular m is a probability measure, this reproduces a result of Henson in [4].

The results of this paper have already been published in my book [13] in somewhat weaker form.

I wish to express here my hearty thanks to F. Niiro, to whom the proof of Lemma 2.2 in $\boldsymbol{R}^+$-valued case is due.

Notations 1) $\mathscr{U}$ is a universe (or a set theoretical structure) which contains all objects to be considered in this paper and ${}^*\mathscr{U}$ is an enlargement of $\mathscr{U}$. We do not assume that ${}^*\mathscr{U}$ is κ-saturated or is sequentially comprehensive. Corresponding notions and objects in ${}^*\mathscr{U}$ are pre-starred, but we shall often omit the asterisk if there is no danger of confusion.

2) If X is a topological space and $a\in X$, we denote by $\mu(a)$ the monad of a and write $\alpha\doteqdot a$ for $\alpha\in\mu(a)$.

3) A field means always a commutative field.

4) $\boldsymbol{N}$ is the set of natural numbers $0, 1, 2, \cdots$. $\boldsymbol{R}$ (resp. $\boldsymbol{C}$) is the set of real (resp. complex) numbers. $\boldsymbol{R}^+=\{a\in\boldsymbol{R};\ a\geqq 0\}$ and $\boldsymbol{R}^{++}=\{a\in\boldsymbol{R};\ a>0\}$.

Part I Linear mappings from a function space

A. Representation by a $*$-finite operator

1.1. LEMMA. *Let X be a set, $\boldsymbol{K}$ a field and $f_1, \cdots, f_n$ a finite number of $\boldsymbol{K}$-valued functions on X. If $f_1, \cdots, f_n$ are linearly independent, then there exist n points $x_1, \cdots, x_n$ in X such that the matrix $(f_i(x_j))_{1\leqq i,j\leqq n}$ is non-singular.*

Proof. For the sake of convenience, we give a simple non-standard proof of the this well known Lemma.

Suppose that the restrictions of $f_1, \cdots, f_n$ to any finite subset of X are linearly dependent. Then, the binary formula $\phi(x, y)$:

$$x \in X \wedge y=(c_1, \cdots, c_n) \in \boldsymbol{K}^n \wedge y \neq 0 \wedge \sum_{i=1}^{n} c_i f_i(x)=0$$

is concurrent in $\mathscr{U}$. Hence there exists a $*$-element $(\gamma_1, \cdots, \gamma_n) \neq (0, \cdots, 0)$ in ${}^*\boldsymbol{K}^n$ satisfying $\sum_{i=1}^n \gamma_i f_i(x)=0$ for all $x \in X$. ${}^*\boldsymbol{K}$ being a linear space over $\boldsymbol{K}$, take a basis $\{\sigma_j; j \in J\}$ of the subspace of ${}^*\boldsymbol{K}$ spanned by $\gamma_1, \cdots, \gamma_n$ and write $\gamma_i=\sum_{j\in J} c_{ij}\sigma_j$. Then, we have for every x in X

$$0=\sum_{i=1}^{n} \gamma_i f_i(x)=\sum_{i=1}^{n} \sum_{j\in J} c_{ij}\sigma_j f_i(x)=\sum_{j\in J}\left[\sum_{i=1}^{n} c_{ij} f_i(x)\right]\sigma_j .$$

Hence we have $\sum_{i=1}^n c_{ij} f_i(x)=0$ for all $j \in J$ and all $x \in X$. Therefore, $c_{ij}=0$ and so $\gamma_1=\cdots=\gamma_n=0$ which contradicts our choice of $(\gamma_1, \cdots, \gamma_n)$.

Consequently, there exists a finite number of points $x_1, \cdots, x_k$ in X such that the matrix $(f_i(x_j))_{1\leq i\leq n, 1\leq j\leq k}$ is of rank n, which completes the proof. ■

1.2. *Definition.* Let $\boldsymbol{K}$ be a topological field and V a topological linear space over $\boldsymbol{K}$. We shall provisionally call V an *HB-space* if the following condition is satisfied: if $v_1, \cdots, v_n$ are a finite number of linearly independent elements in V and $w_1, \cdots, w_n$ are elements in a topological vector space W over $\boldsymbol{K}$, then there exists a continuous linear mapping z from V to W such that $z(v_i)=w_i$ for $1\leqq i\leqq n$.

Examples of *HB*-spaces:

(1) Hausdorff locally convex spaces over $\boldsymbol{R}$ or $\boldsymbol{C}$.

(2) discrete linear spaces over a discrete field $\boldsymbol{K}$.

(3) $\boldsymbol{K}^n$, where $n \in N$ and $\boldsymbol{K}$ is an arbitrary topological field.

1.3. *Notations.* In Part I, we use following notations.

(1) $\boldsymbol{K}$ is a topological field, V is an *HB*-space over $\boldsymbol{K}$, W is a topological linear space over $\boldsymbol{K}$ and Z is the space of all continuous linear mappings from V to W. If $v \in V$ and $z \in Z$, we write $\langle v, z\rangle$ for $z(v)$.

(2) X is a non-empty set and E is a linear space of V-valued functions on X.

(3) We call a function *finite* if its support is a finite set. Denote by F the space of all Z-valued finite functions on X. For every $h \in F$, a linear mapping T_h from E to W is defined by the formula

$$T_h(f) = \sum_{x \in X} \langle f(x), h(x) \rangle$$

for $f \in E$. We shall call this a *finite operator*.

(4) By interpreting this in ${}^*\mathscr{U}$, a $*$-finite operator is defined: for every $\varphi \in {}^*F$, ${}^*T_\varphi\colon {}^*E \rightarrow {}^*W$ is defined by the formula

$${}^*T_\varphi(\psi) = {}^*\sum_{\xi \in {}^*X} \langle \psi(\xi), \varphi(\xi) \rangle$$

for $\psi \in {}^*E$.

(5) Let P be a linear mapping from E to W. Our aim is to represent P by a $*$-finite operator: that is, to find a $\varphi \in {}^*F$ such that ${}^*T_\varphi(f) = P(f)$ or ${}^*T_\varphi(f) \fallingdotseq P(f)$ for all $f \in E$.

1.4. LEMMA. (1) *Notations being as above, for any finite number of elements $f_1, \cdots, f_n$ in E, there exists a finite function h in F such that $T_h(f_i) = P(f_i)$ for $1 \leq i \leq n$.*

(2) *Suppose further that $\boldsymbol{K}$ is not discrete, $V \neq \{0\}$, $W \neq \{0\}$ and let $\mathscr{V}(0)$ be the system of neighborhood of 0 in W (W is not discrete). Then, for any finite number of elements $f_1, \cdots, f_n$ in E, any neighborhood B in $\mathscr{V}(0)$ and for any finite subset A of X, there exists a finite function h in F such that the support of h includes A and that $T_h(f_i) - P(f_i) \in B$ for $1 \leq i \leq n$.*

Proof. The proof of (1) being very similar to and much simpler than that of (2), we shall only prove the latter.

We proceed by induction on n. Note that $Z \neq \{0\}$ because V is an HB-space.

1° $n=1$. (a) If $f=0$, take $z_0 \neq 0$ in Z and put $h(x) = z_0$ for $x \in A$ and $h(x) = 0$ for $x \in X - A$. Then h satisfies the requirement.

(b) If there exists an element $a \in X$ such that $f(a) \neq 0$, take a neighborhood $B' \in \mathscr{V}(0)$ such that $B' + B' \subset B$ and take $w_1 \neq 0$ in W such that $w_1 - P(f) \in B'$. V being an HB-space, there exists $z_1 \neq 0$ in Z such that $\langle f(a), z_1 \rangle = w_1$. Put

$$w_0 = \sum_{x \in A - \{a\}} \langle f(x), z_1 \rangle$$

and take $c \neq 0$ in $\boldsymbol{K}$ such that $cw_0 \in B'$. Define the function h as follows: $h(a) = z_1$, $h(x) = cz_1$ if $x \in A - \{a\}$ and $h(x) = 0$ otherwise. Then h belongs to F, its support includes A and we have

$$\begin{aligned} T_h(f) - P(f) &= \langle f(a), h(a) \rangle - P(f) + \sum_{x \in A - \{a\}} \langle f(x), h(x) \rangle \\ &= (w_1 - P(f)) + cw_0 \in B' + B' \subset B\,. \end{aligned}$$

2° Suppose the Lemma is proved for $n-1$. If $f_1, \cdots, f_n$ are linearly dependent, write $f_n = \sum_{i=1}^{n-1} c_i f_i$ (without loss of generality) and take a neighborhood $B' \in \mathscr{V}(0)$ such that $B' \subset B$ and that $c_1 B' + \cdots + c_{n-1} B' \subset B$. Apply the induction assumption to $f_1, \cdots, f_{n-1}$ and B': there exists a finite function h in F such that the support of h includes A and that $T_h(f_i) - P(f_i) \in B'$ for $1 \leq i \leq n-1$. We have then

$$T_h(f_n) - P(f_n) = \sum_{i=1}^{n-1} c_i [T_h(f_i) - P(f_i)] \in B .$$

3° Suppose $f_1, \cdots, f_n$ are linearly independent. Take an algebraic basis $\{v_k : k \in K\}$ of V and write

$$f_i(x) = \sum_{k \in K} g_i(k, x) v_k \qquad \text{(finite sum for each } x \text{ in } X)$$

with $g_i(k, x) \in \boldsymbol{K}$. Then $g_1, \cdots, g_n$ are linearly independent mappings from $K \times X$ to $\boldsymbol{K}$. In fact, suppose $\sum_{i \in J} c_i g_i = 0$ where $J = \{1, \cdots, n\}$. For a fixed x, there are only finitely many k's such that $g_i(k, x) \neq 0$. Therefore,

$$0 = \sum_{k \in K} [\sum_{i \in J} c_i g_i(k, x)] v_k = \sum_{i \in J} c_i [\sum_{k \in K} g_i(k, x) v_k] = \sum_{i \in J} c_i f_i(x) .$$

Hence we have $c_1 = \cdots = c_n = 0$. By Lemma 1, there exists a finite subset

$$Y = \{y_j = (k_j, x_j);\ j \in J\}$$

with n elements of $K \times X$ such that the matrix $(g_i(y_j))_{1 \leq i, j \leq n}$ is non-singular.

4° Take a basis $\{w_l;\ l \in L\}$ of the subspace of W spanned by $P(f_1), \cdots, P(f_l)$ and write

$$P(f_i) = \sum_{l \in L} P_l(f_i) w_l .$$

For every l in L, the system of linear equations

$$\sum_{j \in J} g_i(y_j) u_j = P_l(f_i) \quad (1 \leq i \leq n)$$

in unknowns $(u_1, \cdots, u_n)$ admits a unique solution $(a_{1,l}, \cdots, a_{n,l})$:

$$\sum_{j \in J} g_i(y_j) a_{j,l} = P_l(f_i) \quad (l \in L,\ i \in J) .$$

5° Put $S = \{x_j;\ j \in J\}$ and, for $x \in S$, put

$$J_x = \{j \in J;\ x_j = x\} .$$

Then $\{J_x;\ x \in S\}$ is a partition of J. If we write

$$f_i(x_j) = \sum_{k \in K} g_i(k, x_j) v_k \quad (i \in J) ,$$

there are only finitely many k's such that $g_i(k, x_j) \neq 0$ for some $(i, j) \in J \times J$. Let K_0 be the set of all such k's:

$$f_i(x_j)=\sum_{k\in K_0} g_i(k, x_j)v_k \quad (i, j\in J)\ .$$

6° As V is an HB-space, there exists, for $j\in J$ and $l\in L$, a continuous linear mapping $z_{j,l}$ from V to W such that

$$\langle v_k, z_{j,l}\rangle=\delta_{k,k_j}w_l$$

for all $k\in K_0$.

7° Let d be the number of elements of $A\cup S$ and take a neighborhood $B'\in\mathscr{V}(0)$ such that

$$\overbrace{B'+\cdots+B'}^{d\text{ times}}\subset B.$$

Define the function h as follows. At first, note that there exists $z_0\neq 0$ in Z such that

$$\langle f_i(x), z_0\rangle\in B'$$

for all $i\in J$ and $x\in A\cup S$ (use the continuity of the mapping $c\mapsto c\langle f_i(x), z\rangle$ from K to W for each i, x and z).

(a) If $x\notin A\cup S$, put $h(x)=0$.

(b) If $x\in A-S$, put $h(x)=z_0$.

(c) If $x\in S$, put first

$$h'(x)=\sum_{j\in J_x}\sum_{l\in L} a_{j,l}z_{j,l}$$

and put

$$h(x)=\begin{cases} h'(x) & \text{if } h'(x)\neq 0\\ z_0 & \text{if } h'(x)=0\ .\end{cases}$$

8° It is clear that h belongs to F and its support includes A. For $i\in J$, we have

$$\begin{aligned} T_h(f_i)&=\sum_{x\in S}\langle f_i(x), h(x)\rangle+\sum_{x\in A-S}\langle f_i(x), h(x)\rangle\\ &=\sum_{x\in S}\langle f_i(x), h'(x)\rangle+\sum_{x\in S}\langle f_i(x), h(x)-h'(x)\rangle+\sum_{x\in A-S}\langle f_i(x), h(x)\rangle\ .\end{aligned}$$

Each term in the second and the third sums belong to B', so that

$$T_h(f_i)-\sum_{x\in S}\langle f_i(x), h'(x)\rangle\in B\ .$$

As for the first sum, we have

$$\begin{aligned}\sum_{x\in S}\langle f_i(x), h'(x)\rangle&=\sum_{x\in S}[\sum_{j\in J_x}\langle f_i(x), \sum_{l\in L} a_{j,l}z_{j,l}\rangle]\\ &=\sum_{x\in S}[\sum_{j\in J_x}\langle f_i(x_j), \sum_{l\in L} a_{j,l}z_{j,l}\rangle]=\sum_{j\in J}\langle f_i(x_j), \sum_{l\in L} a_{j,l}z_{j,l}\rangle\\ &=\sum_{j\in J}\sum_{k\in K_0}\sum_{l\in L} g_i(k, x_j)a_{j,l}\langle v_k, z_{j,l}\rangle\\ &=\sum_{j\in J}\sum_{l\in L} g_i(k_j, x_j)a_{j,l}w_l=\sum_{l\in L} P_l(f_i)w_l=P(f_i)\ .\end{aligned}$$

We have therefore $T_h(f_i)-P(f_i)\in B$ for $1\leq i\leq n$, which completes the proof. ∎

1.5. THEOREM (1) *Notations being as in* 1.3, *there exists a* $*$*-finite function* φ *in* $^*F$ *such that* $^*T_\varphi(f)=P(f)$ *for all* $f\in E$.

(2) *Suppose further that* $\boldsymbol{K}$ *is not discrete,* $V\neq\{0\}$ *and* $W\neq\{0\}$. *Then, there exists a* $*$*-finite function* φ *in* $^*F$ *such that the support of* φ *inludes* X *and that* $^*T_\varphi(f)\doteqdot P(f)$ *for all* $f\in E$.

Proof. We shall only prove (2). Consider the binary formula $\phi(x, h)$:

$$x=(f, a, B)\in E\times X\times\mathscr{V}(0)\wedge h\in F\wedge h(a)\neq 0\wedge T_h(f)-P(f)\in B\,.$$

Lemma 4 implies that $\phi(x, h)$ is concurrent in $\mathscr{U}$. Therefore this is satisfiable in $^*U$, i.e., there exists φ in $^*F$ with required property. ∎

1.6. COROLLARY (Luxemburg [8]) *Let* $\boldsymbol{K}$ *be a topological field and* X *be a topological linear space over* $\boldsymbol{K}$. *Denote by* X' *the space of all continuous linear forms on* X (*we have no concern in topologies on* X'). *Then, for every linear form* P *on* X', *there exists an element in* $^*X$ *such that* $P(f)={}^*f(\xi_P)$ *for all* $f\in X'$.

Proof. In Theorem 1.5, put $V=W=\boldsymbol{K}$, $X=X$ and $E=X'$. Then there exists a $*$-finite function φ in $^*F$ such that $P(f)={}^*T_\varphi(f)$ for all $f\in X'$. Put

$$\xi_P={}^*\sum_{\xi\in{}^*X}\varphi(\xi)\cdot\xi\in{}^*X\,.$$

If f is in X', then $^*f$ is $*$-linear, so we have

$$P(f)={}^*\sum_{\xi\in{}^*X}{}^*f(\xi)\cdot\varphi(\xi)={}^*f\Big({}^*\sum_{\xi\in{}^*X}\varphi(\xi)\cdot\xi\Big)={}^*f(\xi_P)\,.\ ∎$$

1.7. COROLLARY. *Let* $(X, \boldsymbol{B}, m)$ *be a measure space and* E *the space of real-valued integrable functions. If we put* $P(f)=\int_X f\,dm$ *for* $f\in E$, *the theorem can be applied:*

(1) *there exists a* $*$*-finite subset* Γ *of* $^*X$ *and a* $*$*-real valued internal function* φ *on* Γ *such that*

$$\int_X f\,dm={}^*\sum_{\xi\in\Gamma}{}^*f(\xi)\varphi(\xi)$$

for every integrable function f *on* X.

(2) *there exists a* $*$*-finite subset* Γ *of* $^*X$ *including* X *and a* $*$*-real valued nowhere vanishing internal function* φ *on* Γ *such that*

$$\int_X f\,dm={}^*\sum_{\xi\in\Gamma}{}^*f(\xi)\varphi(\xi)$$

for every integrable function f *on* X.

Remark. The latter result will be generalized later to not necessarily integrable functions having a definite integral value in $\boldsymbol{R}\cup\{+\infty, -\infty\}$.

B. Representation by $*$-integration

1.8. *Notations and assumptions.* We follow the notations in 1.3 and suppose $\boldsymbol{K}$ is $\boldsymbol{R}$ or $\boldsymbol{C}$ and V, W are non-zero Banach spaces over $\boldsymbol{K}$. Suppose further that X is a Hausdorff locally compact space and that the elements of E are continuous functions from X to V.

Let m be an everywhere strictly positive Radon measure on X; i.e., $m(A)>0$ for every compact neighborhood in X.

1.9. LEMMA. *Notations being as above, for any finite number of functions $f_1, \cdots, f_n$ in E, any strictly positive real number $e\in\boldsymbol{R}^{++}$ and for any finite subset D of X, there exist a finite-dimensional subspace Z_0 of Z (Z is also a non-zero Banach space) and a Z_0-valued continuous function g on X whose support is compact and includes D such that*

$$||m(\langle f_i, g\rangle)-P(f_i)||\leq e$$

for $1\leq i\leq n$, where $\langle f_i, g\rangle$ is a W-valued function on X: $x\mapsto\langle f_i(x), g(x)\rangle$.

If in particular X is a C^∞-manifold, then g can be taken as a C^∞-function (this makes sense because Z_0 is finite-dimensional).

Proof. 1° By Lemma 1.4, there exists a finite function h in F whose support includes D such that

$$||T_h(f_i)-P(f_i)||\leq\frac{e}{2}$$

for $1\leq i\leq n$. Let $C=\{a_1, \cdots, a_r\}$ be the support of h. For each $s(1\leq s\leq r)$, take a compact neighborhood A_s of a_s such that $A_s\cap A_{s'}=\emptyset$ if $s\neq s'$. Put $A=A_1\cup\cdots\cup A_r$.

2° It suffices to show that, for each $s(1\leq s\leq r)$, there exists a continuous (resp. C^∞) function g_s on X with following properties:

(1) the range of g_s is included in a one-dimensional subspace of Z.

(2) the support of g_s is included in A_s and $g_s(a_s)\neq 0$.

(3) $||m(\langle f_i, g_s\rangle)-\langle f_i(a_s), h(a_s)\rangle||\leq e/2r$

for $1\leq i\leq n$.

In fact, if this is done, the function $g=\sum_{s=1}^r g_s$ is continuous (resp. C^∞), the support of g is compact and includes D, the range of g is included in a finite-dimensional subspace of Z and we have

$$\begin{aligned}||m(\langle f_i, g\rangle)-P(f_i)||&\leq||m(\langle f_i, g\rangle)-T_h(f_i)||+||T_h(f_i)-P(f_i)||\\&\leq\sum_{s=1}^r||m(\langle f_i, g_s\rangle)-\langle f_i(a_s), h(a_s)\rangle||+\frac{e}{2}\leq e\,,\end{aligned}$$

which will complete the proof.

3° Take a small compact neighborhood C_s of a_s included in the interior A_s° of A_s such that

$$\|f_i(x)-f_i(a_s)\| \leq \frac{e}{4r\|h(a_s)\|}$$

for all $x \in C_s$. We have $m(C_s)>0$ by the assumption. Put

$$M=1+\max\{\|f_i(x)\|;\ 1\leq i\leq n,\quad x\in A\}\ .$$

There exists a compact neighborhood B_s of a_s such that

$$C_s \subset B_s^\circ \subset B_s \subset A_s^\circ\ .$$

and that

$$m(B_s-C_s) \leq \frac{m(C_s)e}{4rM\|h(a_s)\|}\ .$$

4° The one-dimensional subspace Z_s of Z spanned by $h(a_s)$ is isomorphic to $\boldsymbol{K}$. Hence there exists a Z_s-valued continuous (resp. C^∞) function g_s on X with following properties:

(a) the support of g_s is included in B_s.

(b) If $x\in C_s$, then $g_s(x)=h(a_s)/m(C_s)$.

(c) $\|g_s(x)\|\leq\|h(a_s)\|/m(C_s)$ for all $x\in X$.

5° For every $i(1\leq i\leq n)$, we have

$$\begin{aligned}
&\|m(\langle f_i, g_s\rangle)-\langle f_i(a_s), h(a_s)\rangle\| \\
&\quad=\left\|\int_{C_s}\langle f_i(x),\ g_s(x)\rangle dm(x)+\int_{B_s-C_s}\langle f_i(x),\ g_s(x)\rangle dm(x)\right. \\
&\qquad\left.-\int_{C_s}\langle f_i(a_s),\ g_s(a_s)\rangle dm(x)\right\| \\
&\quad\leq\|g_s(a_s)\|\int_{C_s}\|f_i(x)-f_i(a_s)\|dm(x)+M\cdot\frac{\|h(a_s)\|}{m(C_s)}\cdot m(B_s-C_s) \\
&\quad\leq\|g_s(a_s)\|\cdot\frac{e}{4r\|h(a_s)\|}\cdot m(C_s)+\frac{e}{4r}=\frac{e}{2r}\ .
\end{aligned}$$

We have therefore proved the Lemma. ■

1.10. THEOREM. *Notations and assumptions being as in* 1.8, *there exists a $*$-continuous internal function φ from $^*X$ to $^*Z$ with following properties*:

(a) *the range of φ is included in a $*$-finite-dimensional subspace of $^*Z$,*

(b) *the support of φ is $*$-compact and includes X.*

(c) $^*m(\langle {}^*f, \varphi\rangle)\fallingdotseq P(f)$ *for all* $f\in E$.

If in particular X is a C^∞-manifold, φ can be taken as a $^*C^\infty$-function.

Proof. Denote by C_0(resp. C_0^∞) the set of continuous (resp. C^∞)

functions from X to Z with compact support and with range included in a finite-dimensional subspace of Z. Consider the binary formula $\phi(x, g)$:

$$x=(f, e, a)\in E\times \boldsymbol{R}^{++}\times X\wedge g\in C_0(\text{resp. } C_0^\infty)\wedge g(a)\neq 0\wedge$$
$$\|m(\langle f_i, g\rangle)-P(f)\|\leq e .$$

By Lemma 1.9, $\phi(x, g)$ is concurrent in $\mathscr{U}$. Hence there exists a $*$-function φ with required properties. ■

Remark. Theorem 1.10 applies in particular to the case where $X=\boldsymbol{R}^k$, $V=W=\boldsymbol{R}$ or $\boldsymbol{C}$, m is the Lebesgue measure and P is a generalized function in the sense of Gelfand-Šilov, especially to Schwartz distributions.

1.11. THEOREM (Robinson [12]) *Notations and assumptions being as in* 1.8, *suppose* $X=\boldsymbol{R}^k(k\in \boldsymbol{N})$, $V=W=\boldsymbol{R}$ *or* $\boldsymbol{C}$ *and* m *is the Lebesgue measure on* $\boldsymbol{R}^k$. *If all functions in* E *are of compact support, there exists a* $*$*-polynomial* (resp. $*$*-Fourier polynomial*) φ *in* k *variables and with coefficients in* $^*\boldsymbol{K}$ *such that*

$$P(f)\doteqdot {}^*\!\!\int_{^*\boldsymbol{R}^k} {}^*f(\xi)\varphi(\xi)^*d\xi$$

for all $f\in E$. *We can take* φ *with* $\varphi(x)\neq 0$ *for all* $x\in \boldsymbol{R}^k$.

Proof. 1° Return to Lemma 1.8. Following notations in Lemma, $f_1, \cdots, f_n$ and g are of compact support. Let K be the union of supports of $f_1, \cdots, f_n$ and g, and put

$$v=\int_K 1\cdot dx>0 ,$$
$$M=1+\max\{|f_i(x)|;\ 1\leq i\leq n,\quad x\in K\} .$$

By the approximation theorem of Weierstrass, there exists for every $e\in \boldsymbol{R}^{++}$ a polynomial (resp. Fourier polynomial) p in k variables and with coefficients in $\boldsymbol{K}$ such that $p(x)\neq 0$ for $x\in D$ and that

$$|g(x)-p(x)|\leq\frac{e}{2Mv}$$

for all $x\in K$. We have therefore

$$\left|\int_{\boldsymbol{R}^k} f_i(x)g(x)dx-\int_{\boldsymbol{R}^k} f_i(x)p(x)dx\right|\leq\frac{e}{2}$$

and

$$\left|\int_{\boldsymbol{R}^k} f_i(x)p(x)dx-P(f_i)\right|\leq e$$

for $1\leq i\leq n$.

2° Let S be the set of polynomials (resp. Fourier polynomials) in

k variables and with coefficients in $\boldsymbol{K}$ and consider the binary formula $\phi(x, p)$:

$$x=(f, e, a) \in E\times \boldsymbol{R}^{++}\times \boldsymbol{R}^k \wedge p \in S \wedge p(a)\neq 0 \wedge \left| \int_{\boldsymbol{R}^k} f(x)p(x)dx-P(f) \right| \leq e .$$

By the consideration above, $\phi(x, p)$ is concurrent in $\mathscr{U}$. Hence there exists a $*$-polynomial (resp. $*$-Fourier polynomial) φ with required properties. ∎

Part II Conic forms and measures

A. Representation by a $*$-finite operator with positive kernel

2.1. *Notations and Definitions.* In Part II, we use following notations and conventions.

(1) $\overline{\boldsymbol{R}}=\boldsymbol{R}\cup\{-\infty, +\infty\}$ is the completed real line with usual order relation. This is a Hausdorff compact space in the order topology. The monad $\mu(+\infty)$ (resp. $\mu(-\infty)$) is the set of $+\infty$ (resp. $-\infty$) and of all positive (resp. negative) infinite $*$-real numbers. So we write $\alpha \fallingdotseq +\infty$ (resp. $\alpha \fallingdotseq -\infty$) if $\alpha \in \mu(+\infty)$ (resp. $\alpha \in \mu(-\infty)$).

(2) We define some algebraic operations including $\pm\infty$: $(\pm\infty)+(\pm\infty)=\pm\infty$. For $a \in \boldsymbol{R}$, $a+(\pm\infty)=(\pm\infty)+a=\pm\infty$. For $a \in \boldsymbol{R}^{++}$, $a\cdot(\pm\infty)=(\pm\infty)\cdot a=\pm\infty$. We do not define $0\cdot(\pm\infty)$.

(3) X is a set. E is a *conic space* of $\boldsymbol{R}^+$-valued functions on X. That is, $0 \in E$ and $af+bg \in E$ whenever $f, g \in E$ and $a, b \in \boldsymbol{R}^+$.

(4) P is a *conic form* on E. That is, P is a mapping from E to $\overline{\boldsymbol{R}}^+=\{x \in \overline{\boldsymbol{R}};\ x\geq 0\}$ with following properties (operations, equalities and inequalities in the right-hand side should be understood in $\overline{\boldsymbol{R}^+}$):

(a) $P(0)=0$.

(b) $a \in \boldsymbol{R}^{++}, f \in E \Rightarrow P(af)=aP(f)$,

(c) $f, g \in E \Rightarrow P(f+g)=P(f)+P(g)$,

(d) $f, g \in E,\ f\leq g \Rightarrow P(f)\leq P(g)$.

Here, $f\leq g$ means $f(x)\leq g(x)$ for all $x \in X$.

(5) Put $E_P=\{f \in E;\ P(f)<+\infty\}$. Note that $0 \in E_P$.

(6) F is the set of $\boldsymbol{R}^+$-valued finite functions on X. For $h \in F$, an $\boldsymbol{R}^+$-valued conic form T_h on E is defined by the formula

$$T_h(f)=\sum_{x \in X} f(x)h(x) \quad (f \in E) .$$

2.2. LEMMA. *Let $f_1, \cdots, f_l \in E_P, f_{l+1}, \cdots, f_n \in E-E_P, e \in \boldsymbol{R}^{++}, N \in \boldsymbol{N}$ and D a finite subset of X. There exists a positive finite function h in F whose support includes D such that*

$$|P(f_i)-T_h(f_i)|\leq e(1\leq i\leq l) ,$$
$$T_h(f_i)\geq N \quad (l+1\leq i\leq n) .$$

Proof. 1° Put $\tilde{E}_P=\{f-g;\ f, g \in E_P\}$. Then $\tilde{E}_P$ is a linear space and P can be extended to a positive linear form on $\widetilde{E}_P$ by putting $P(f-g)=P(f)-P(g)$. Let V be the finite-dimensional subspace of $\widetilde{E}_P$ spanned by $f_1, \cdots, f_l$. Put $f_0=f_1+\cdots+f_l$. Then $f_0 \in E_P$. Let X_0 be the support of f_0, denote by d the number of elements of D and put

$$M=1+\max\{f_i(x);\ 1\leq i\leq n,\quad x\in D\}\ .$$

2° If $P(f_0)=0$, let $h_0(x)=e/2dM$ for $x\in D$ and $h_0(x)=0$ for $x\in X-D$. Then h_0 belongs to F, its support includes D and

$$|P(f_i)-T_{h_0}(f_i)|\leq \sum_{x\in D} f_i(x)h_0(x)\leq \frac{e}{2}$$

for $1\leq i\leq l$.

3° Suppose $P(f_0)>0$. Define a mapping $\boldsymbol{f}$ from the dual space V' of V to $\boldsymbol{R}^l$ by putting

$$\boldsymbol{f}(s)=(s(f_1),\ \cdots,\ s(f_l))$$

for $s\in V'$. If s is a positive linear form on V, then $\boldsymbol{f}(s)_i=s(f_i)\geq 0$.

For $x\in X_0$, an element $s_x\in V'$ is defined by

$$s_x(f)=\frac{f(x)}{f_0(x)}\quad (f\in V)\ .$$

We have

$$\boldsymbol{f}(s_x)=(s_x(f_1),\ \cdots,\ s_x(f_l))=\left(\frac{f_1(x)}{f_0(x)},\ \cdots,\ \frac{f_l(x)}{f_0(x)}\right)\ .$$

4° Denote by the same symbol P the restriction of P (extended to $\widetilde{E}_P$) to V. Then $P\in V'$. We claim that $\boldsymbol{f}(P/P(f_0))$ belongs to the closed convex hull C of $\{\boldsymbol{f}(s_x);\ x\in X_0\}$ in $\boldsymbol{R}^l$. In fact, if the claim is not true, the point $\boldsymbol{f}(P/P(f_0))$ and the set C is strictly separated by a hyperplane: there exist real numbers $b_0, b_1, \cdots, b_l$ such that

$$\sum_{i=1}^{l} b_i\frac{P(f_i)}{P(f_0)}>b_0>\sup_{x\in X_0}\sum_{i=1}^{l} b_i\frac{f_i(x)}{f_0(x)}\ .$$

Put $g=\sum_{i=1}^{l} b_i f_i$. Then $g\in V$ and

$$\frac{P(g)}{P(f_0)}>b_0>\sup_{x\in X_0}\frac{g(x)}{f_0(x)}\ .$$

Therefore $g(x)<b_0f_0(x)$ for all $x\in X_0$. If $x\in X-X_0$, the both sides are zero, and hence we have $g\leq b_0 f$. We have therefore $P(g)\leq b_0P(f_0)$, that is, $P(g)/P(f_0)\leq b_0$, which is absurd.

5° Hence there exist a finite number of points $x_1, \cdots, x_m$ in X_0 and elements $a_1, \cdots, a_m$ in $\boldsymbol{R}^{++}$ such that $\sum_{j=1}^{m} a_j=1$ and that

$$\left\|\boldsymbol{f}\left(\frac{P}{P(f_0)}\right)-\sum_{j=1}^{m} a_j\boldsymbol{f}(s_{x_j})\right\|\leq\frac{e}{4P(f_0)}\ ,$$

where $\|(y_1, \cdots, y_l)\| = \max_{1\leq i\leq l} |y_i|$ for $(y_1, \cdots, y_l) \in \boldsymbol{R}^l$. It follows that

$$\left|\frac{P(f_i)}{P(f_0)} - \sum_{j=1}^{m} a_j \frac{f_i(x_j)}{f_0(x_j)}\right| \leq \frac{e}{4P(f_0)}$$

for $1\leq i\leq l$, and that

$$\left|P(f_i) - \sum_{j=1}^{m} \frac{a_j P(f_0)}{f_0(x_j)} f_i(x_j)\right| \leq \frac{e}{4}$$

for $1\leq i\leq l$.

6° Define a finite function h_0' in F by putting $h_0'(x_j) = a_j P(f_0)/f_0(x_j)$ for $1\leq j\leq m$ and $h_0'(x)=0$ otherwise.

Define a finite function h_0'' in F by putting $h_0''(x)=e/4dM$ for $x\in D$ and $h_0''(x)=0$ for $x\in X-D$. We have $T_{h_0''}(f_i)\leq e/4$ for $1\leq i\leq n$.

Put $h_0=h_0'+h_0''$. Then h_0 belongs to F and its support includes D.

We have, for $1\leq i\leq l$,

$$\begin{aligned} |T_{h_0}(f_i)-P(f_i)| &= \left|\sum_{x\in X} f_i(x)h_0'(x) + \sum_{x\in X} f_i(x)h_0''(x) - P(f_i)\right| \\ &\leq \left|\sum_{x\in X} f_i(x)h_0'(x) - P(f_i)\right| + \sum_{x\in D} f_i(x)h_0''(x) \\ &\leq \left|\sum_{j=1}^{m} f_i(x_j)\frac{a_j P(f_0)}{f_0(x_j)} - P(f_i)\right| + \frac{e}{4} \leq \frac{e}{4} + \frac{e}{4} = \frac{e}{2}\,. \end{aligned}$$

We have thus shown that the function h_0 satisfies the requirement of Lemma, except for last inequalities $T_h(f_i)\geq N(l+1\leq i\leq n)$, regardless of the vanishing of $P(f_0)$.

7° If $l=n$, Lemma has already been proved. We suppose $l<n$ and $N\geq 1$. Assume without loss of generality that the supports of $f_{l+1}, \cdots, f_m$ are not in X_0 and those of $f_{m+1}, \cdots, f_n$ are in X_0.

(a) For $k(l+1\leq k\leq m)$, take $y_k\in X-X_0$ such that $f_k(y_k)>0$ and define a function h_k in F by putting $h_k(y_k)=N/f_k(y_k)$ and $h_k(y)=0$ otherwise.

(b) For $k(m+1\leq k\leq n)$, put $N'=4(n-l)N/e$. $P(f_k)=+\infty$ implies $f_k\notin N'f_0$, so there exists a point $y_k\in X_0$ such that $f_k(y_k)>N'f_0(y_k)$. Define a function h_k in F by putting $h_k(y_k)=N/N'f_0(y_k)=e/4(n-l)f_0(y_k)$ and $h_k(y)=0$ otherwise.

8° Put $h=h_0+\sum_{k=l+1}^{n} h_k$. Then h belongs to F and its support includes D. If $1\leq i\leq l$, we have

$$\begin{aligned} |T_h(f_i)-P(f_i)| &\leq |T_{h_0}(f_i)-P(f_i)| + \sum_{k=l+1}^{n} T_{h_k}(f_i) \\ &\leq \frac{e}{2} + \sum_{k=l+1}^{n} f_i(y_k)h_k(y_k) \leq \frac{e}{2} + \sum_{k=m+1}^{n} f_0(y_k)h_k(y_k) \leq \frac{e}{2} + \frac{e}{2} = e\,. \end{aligned}$$

For $l+1\leq k\leq n$, we have $T_h(f_k)\geq T_{h_k}(f_k)=f_k(y_k)h_k(y_k)$. If $l+1\leq k\leq m$, then $f_k(y_k)h_k(y_k)=N$. If $m+1\leq k\leq n$, then

$$f_k(y_k)h_k(y_k) > N'f_0(y_k)\frac{N}{N'f_0(y_k)} = N .$$

Hence we have $T_h(f_k) \geq N$ for $l+1 \leq k \leq n$, which completes the proof of Lemma. ■

2.3. THEOREM. *Notations and assumptions being as in* 2.1, *there exist a $*$-finite subset Γ of $^*X$ including X and a $^*\boldsymbol{R}^{++}$-valued internal function φ on Γ such that*

$$P(f) \doteqdot {}^*\sum_{\xi \in \Gamma} {}^*f(\xi)\varphi(\xi)$$

for all $f \in E$.

Proof. Consider the binary formula $\phi(x, h)$:

$$x = (f, e, N, a) \in E \times \boldsymbol{R}^{++} \times \boldsymbol{N} \times X \wedge h \in F \wedge h(a) \neq 0 \wedge$$
$$[f \in E_P \rightarrow |P(f) - T_h(f)| \leq e] \wedge [f \in E - E_P \rightarrow T_h(f) \geq N] .$$

By the Lemma, $\phi(x, h)$ is concurrent in $\mathscr{U}$. Hence there exists a $*$-finite function φ in $^*F$ whose support includes X such that $^*T_\varphi(^*f) \doteqdot P(f)$ for $f \in E_P$ and that $^*T_\varphi(^*f) \doteqdot +\infty = P(f)$ for $f \in E - E_P$. ■

2.4. *Definition.* Let $(X, \boldsymbol{B}, m)$ be a measure space. If f is a measurable $\boldsymbol{R}$-valued function on X, we put

$$f^+(x) = \max\{f(x), 0\},\ f^-(x) = \max\{-f(x), 0\} .$$

Then f^+ and f^- are measurable and $f = f^+ - f^-$. If one of f^+, f^- is integrable, we can assign a definite integral value in $\overline{\boldsymbol{R}}$:

$$\int_X f\,dm = \int_X f^+\,dm - \int_X f^-\,dm .$$

In this case, we shall say that f *has a definite integral value.*

2.5. THEOREM. *Let $(X, \boldsymbol{B}, m)$ be a measure space. Then, there exist a $*$-finite subset Γ of $^*X$ including X and a $^*\boldsymbol{R}^{++}$-valued internal function φ on Γ such that*

$$\int_X f\,dm \doteqdot {}^*T_\varphi(^*f) = {}^*\sum_{\xi \in \Gamma} {}^*f(\xi)\varphi(\xi) \quad (\text{in } {}^*\overline{\boldsymbol{R}})$$

for every function f having a definite integral value.

Proof. Let E be the set of all $\boldsymbol{R}^+$-valued measurable functions and put

$$P(f) = \int_X f\,dm \in \overline{\boldsymbol{R}}^+$$

for $f \in E$. Then the Theorem 2.3 applies. For a function f having a definite integral value, it suffices to write $f = f^+ - f^-$ and apply the Theorem to f^+ and f^-. ■

2.6. *Definition.* If, in the definition of a measure space $(X, \boldsymbol{B}, m)$, we assume only that $\boldsymbol{B}$ is an algebra of subsets of X and that m is finitely additive, we call this a *quasi-measure space.*

2.7. THEOREM. *Let $(X, \boldsymbol{B}, m)$ be a quasi-measure space.*

(1) *There exists a ${}^*\boldsymbol{R}^+$-valued internal function φ on X with $*$-finite support including X such that*

$$m(A) \doteqdot {}^*\sum_{\xi \in {}^*A} \varphi(\xi)$$

for all $A \in \boldsymbol{B}$.

(2) *Put $\tilde{m}_\varphi(A) = st[{}^*\sum_{\xi \in {}^*A} \varphi(\xi)] \in \bar{\boldsymbol{R}}^+$ for $A \subset X$, where $st[\alpha]$ is the standard part of $\alpha \in {}^*\bar{\boldsymbol{R}}^+$. Then $\tilde{m}_\varphi$ is a totally defined quasi-measure on X and $\tilde{m}_\varphi(A) = m(A)$ for all $A \in \boldsymbol{B}$.*

Proof. Let E be the set of all measurable simple functions. For every $f \in E$, write

$$f = \sum_{j=1}^{p} a_j d_{A_j} \quad (a_j \in \boldsymbol{R}^{++},\ A_j \in \boldsymbol{B})$$

(d_{A_j} is the characteristic function of A_j) and put

$$P(f) = \sum_{j=1}^{p} a_j m(A_j) \ .$$

Then the Theorem 2.3 applies. ∎

B. Representation by a $*$-finite operator with constant kernel

2.8. *Notations.* Let d_A be the characteristic function of a subset A of X. For a finite subset C of X and for $r \in \boldsymbol{R}^{++}$, we write $T_{C,r}$ for $1/r \cdot T_{d,C}$: $T_{C,r}(f) = 1/r \cdot \sum_{x \in C} f(x)$. Interpreting this in ${}^*\mathscr{U}$, for a $*$-finite subset Γ of ${}^*X$ and for $\rho \in {}^*\boldsymbol{R}^{++}$,

$${}^*T_{\Gamma,\rho}(\psi) = \frac{1}{\rho} {}^*\sum_{\xi \in \Gamma} \psi(\xi) \ .$$

2.9. LEMMA. *Notations and assumptions being as in Lemma 2.2, suppose that X is a Hausdorff space without isolated point and that all functions in E are continuous. Then the function h can be chosen as a constant multiple of the characteristic function of a finite subset of X. More precisely, there exist a finite subset C of X including D and a natural number $r \in \boldsymbol{N}$ such that*

$$|P(f_i) - T_{C,r}(f_i)| = \left| P(f_i) - \frac{1}{r} \sum_{x \in C} f_i(x) \right| \le e \quad (1 \le i \le l) \ ,$$

$$T_{C,r}(f_i) = \frac{1}{r} \sum_{x \in C} f_i(x) \ge N \quad (l+1 \le i \le n) \ .$$

Proof. 1° Let h be the finite function in Lemma 2.2, $\{a_1, \cdots, a_k\}$

the support of h and $M=1+\max_{1\leq i\leq n}\sum_{j=1}^{k} f_i(a_j)$. Approximate positive real numbers $h(a_1), \cdots, h(a_k)$ by rational numbers: there exist natural numbers $r, q_1, \cdots, q_k \in N$ such that

$$\left|h(a_j)-\frac{q_j}{r}\right|\leq\frac{e}{2M}$$

for $1\leq j\leq k$. Put $q=q_1+\cdots+q_k$. X being Hausdorff and $f_1, \cdots f_n$ being continuous, there exist mutually disjoint neighborhoods A_j of $a_j (1\leq j\leq k)$ such that

$$|f_i(a_j)-f_i(x)|\leq\frac{re}{2q}$$

for $x\in A_j (1\leq i\leq n, 1\leq j\leq k)$. As a_j is not isolated, A_j is an infinite set. Take q_j elements including a_j from A_j:

$$x_{j,1}=a_j, \quad x_{j,2}, \cdots, x_{j,q_j}.$$

Let C be the set of all these points. C consists of q elements and includes D.

2° We have, for $1\leq i\leq n$,

$$\begin{aligned}|T_{C,r}(f_i)-T_h(f_i)| &= \left|\frac{1}{r}\sum_{j=1}^{k}\sum_{p=1}^{q_j} f_i(x_{j,p})-\sum_{j=1}^{k} f_i(a_j)h(a_j)\right| \\ &\leq \left|\frac{1}{r}\sum_{j=1}^{k}\sum_{p=1}^{q_j} f_i(x_{j,p})-\frac{1}{r}\sum_{j=1}^{k} q_j f_i(a_j)\right|+\left|\frac{1}{r}\sum_{j=1}^{k} q_j f_i(a_j)-\sum_{j=1}^{k} f_i(a_j)h(a_j)\right| \\ &\leq \frac{1}{r}\sum_{j=1}^{k}\sum_{p=1}^{q_j}|f_i(x_{j,p})-f_i(a_j)|+\sum_{j=1}^{k} f_i(a_j)\left|\frac{q_j}{r}-h(a_j)\right| \\ &\leq \frac{q}{r}\cdot\frac{er}{2q}+\frac{e}{2M}\sum_{j=1}^{k} f_i(a_j)\leq e.\end{aligned}$$

Therefore, if $i\leq l$, we have

$$|T_{C,r}(f_i)-P(f_i)|\leq|T_{C,r}(f_i)-T_h(f_i)|+|T_h(f_i)-P(f_i)|\leq 2e$$

and if $i>l$, we have

$$T_{C,r}(f_i)\geq T_h(f_i)-e\geq N-e,$$

which completes the proof of Lemma. ■

2.10. THEOREM. *Notations and assumptions being as in Theorem 2.3, suppose that X is a Hausdorff space without isolated points and that all functions in E are continuous. Then, there exist a $*$-finite subset Γ of ${}^*X$ including X and a $*$-natural number $\rho\in{}^*N$ such that*

$$P(f)\doteqdot{}^*T_{\Gamma,\rho}({}^*f)=\frac{1}{\rho}\,{}^*\sum_{\xi\in\Gamma} f(\xi)$$

for all $f\in E$.

Proof. Denote by $\mathscr{S}(X)$ the set of all finite subset of X and consider the binary formula $\phi(x, y)$:

$$x=(f, e, N, a) \in E \times \boldsymbol{R}^{++} \times \boldsymbol{N} \times X \wedge y=(C, r) \in \mathscr{S}(X) \times \boldsymbol{N} \wedge$$
$$a \in C \wedge [f \in E_P \rightarrow |P(f)-T_{C,r}(f)| \leq e] \wedge [f \in E-E_P \rightarrow T_{C,r} \geq N] .$$

Lemma 2.9 implies that $\phi(x, y)$ is concurrent in $\mathscr{U}$. Hence there exist a $*$-finite subset Γ of ${}^*X$ and $\rho \in {}^*\boldsymbol{N}$ with required properties. ■

2.11. COROLLARY. *Let X be a Hausdorff locally compact space without isolated point and m a positive Radon measure on X. Then, there exist a $*$-finite subset Γ of ${}^*X$ including X and a $*$-natural number $\rho \in {}^*\boldsymbol{N}$ such that*

$$\int_X f\, dm \fallingdotseq {}^*T_{\Gamma,\rho}({}^*f) \frac{1}{\rho}\, {}^*\sum_{\xi \in \Gamma} {}^*f(\xi)$$

for every $\boldsymbol{R}$-valued continuous function f having a definite integral value.

Proof. Let E be the set of all $\boldsymbol{R}^+$-valued continuous functions and put

$$P(f)=\int_X f\, dm \in \overline{\boldsymbol{R}}^+$$

for $f \in E$. Then the Theorem 2.10 applies. If f is not positive, write $f=f^+-f^-$. Then f^+, f^- belong to E and it suffices to apply the Corollary to f^+ and f^-. ■

Part III Non-atomic measure

3.1. *Definition.* Let $(X, \boldsymbol{B}, m)$ be a measure space or a quasi-measure space. If $m(A)=0$ whenever A is measurable and finite, we call m *non-atomic.*

3.2. LEMMA. *Let $(X, \boldsymbol{B}, m)$ be a non-atomic measure space, E the set of positive measurable functions and E_P the set of positive integrable functions. Write $P(f)=\int_X f\, dm$ for $f \in E$. Suppose we are given $f_1, \cdots, f_l \in E_P$, $f_{l+1}, \cdots, f_n \in E-E_P$, $e \in \boldsymbol{R}^{++}$, $N \in \boldsymbol{N}$ and D a finite subset of X. Then, there exist a finite subset C of X including D and a natural number $r \in \boldsymbol{N}$ such that*

$$|T_{C,r}(f_i)-P(f_i)| \leq e \quad (1 \leq i \leq l) ,$$
$$T_{C,r}(f_i) \geq N \quad (l+1 \leq i \leq n)$$

where $T_{C,r}(f)=1/r \cdot \sum_{x \in C} f(x)$ for $f \in E$.

Proof. Assume $e \leq 1 \leq N$ and put

$$M=1+|D| \cdot \max\{f_i(x);\ 1 \leq i \leq n,\quad x \in D\}$$

where $|D|$ is the cardinality of D.

1° By the definition of integral, there exists, for $1 \leq i \leq l$, a positive integrable simple function g_i such that $g_i \leq f_i$ and that $P(f_i) - P(g_i) \leq 1/8 \cdot e$. And for $l+1 \leq i \leq n$, there exists a positive integrable simple function g_i such that $g_i \leq f_i$ and $P(g_i) \geq 2N$. Put $G = \max\{g_i(x);\ 1 \leq i \leq n,\ x \in X\}$ and let Y be the union of the supports of $g_1, \cdots, g_n$. Then $m(Y) < +\infty$.

2° There exists a measurable subset Z of Y such that $m(Y-Z) < e/8G$ and that $f_1, \cdots, f_l$ are bounded on Z. Put

$$f_i'(x) = \begin{cases} f_i(x) & \text{if } x \in Z \\ 0 & \text{if } x \notin Z \end{cases}$$

for $1 \leq i \leq l$. Then $f_1', \cdots, f_l'$ are bounded and $g_i(x) \leq f_i'(x)$ for $x \in Z$, so that there exists a positive integrable simple function $h_i (1 \leq i \leq l)$ such that $h_i \leq f_i'$, $g_i(x) \leq h_i(x)$ for $x \in Z$ and that

$$f_i'(x) - h_i(x) < \frac{e}{8(1+m(Y))}$$

for $x \in X$. If $i > l$, put simply

$$h_i(x) = \begin{cases} g_i(x) & \text{if } x \in Z \\ 0 & \text{if } x \notin Z\,. \end{cases}$$

3° There exists a finite disjoint family $\{A_1, \cdots, A_p\}$ of measurable subsets of Z such that $h_1, \cdots, h_n$ are positive linear combinations of characteristic functions d_{A_j} of $A_j (1 \leq j \leq p)$:

$$h_i = \sum_{j=1}^{p} a_{ij} d_{A_j} \quad (a_{ij} \geq 0,\ 1 \leq i \leq n)\,.$$

Put $a_0 = 1 + \max_{1 \leqq i \leqq n} \sum_{j=1}^{p} a_{ij}$. Approximate real numbers $m(A_1), \cdots, m(A_p)$ by rational numbers: there exist natural numbers $r, q_1, \cdots, q_p$ in N such that $1/r \leq e/4M$ and that

$$\frac{q_j}{r} \leq m(A_j) \leq \frac{q_j}{r} + \frac{e}{8a_0}$$

for $1 \leq j \leq p$. If we write $q = q_1 + \cdots + q_r$, we have

$$\frac{q}{r} \leq \sum_{j=1}^{p} m(A_j) \leq m(Z) \leq m(Y)\,.$$

4° Take q_j points from A_j. This is possible because A_j is infinite if $m(A_j) > 0$ and $q_j = 0$ if $m(A_j) = 0$. Let C_0 be the set of all these points $(|C_0| = q)$ and put $C = C_0 \cup D$.

For $1 \leq i \leq n$, we have

$$\begin{aligned}T_{C,r}(h_i) &= T_{C_0,r}(h_i)+T_{D-C_0,r}(h_i)\\ &=\frac{1}{r}\sum_{j=1}^{p}\sum_{x\in C_0\cap A_j} h_i(x)+\frac{1}{r}\sum_{x\in D-C_0} h_i(x)\\ &=\frac{1}{r}\sum_{j=1}^{p} a_{ij}q_j+\frac{1}{r}\sum_{x\in D-C_0} h_i(x)\,.\end{aligned}$$

This equality together with $P(h_i)=\sum_{j=1}^{p} a_{ij}m(A_j)$ implies that

$$\begin{aligned}|T_{C,r}(h_i)-P(h_i)| &\le |T_{C_0,r}(h_i)-P(h_i)|+T_{D-C_0,r}(h_i)\\ &\le \sum_{j=1}^{p} a_{ij}\left|\frac{q_j}{r}-m(A_j)\right|+\frac{1}{r}\sum_{x\in D-C_0} h_i(x)\le\frac{e}{8}+\frac{e}{4}\,.\end{aligned}$$

5° If $1\le i\le l$, we have

$$\begin{aligned}0 &\le P(f_i)-P(h_i)=(P(f_i)-P(g_i))+(P(g_i)-P(h_i))\\ &\le\frac{e}{8}+\int_Z(g_i-h_i)+\int_{Y-Z} g_i\le\frac{e}{8}+0+G\cdot\frac{e}{8G}=\frac{e}{4}\,.\end{aligned}$$

On the other hand, we have

$$0\le T_{C,r}(f_i)-T_{C,r}(h_i)=\frac{1}{r}\sum_{x\in C_0}(f_i'(x)-h_i(x))+\frac{1}{r}\sum_{x\in D-C_0}(f_i(x)-h_i(x))$$
$$\le\frac{q}{r}\,\frac{e}{8(1+m(Y))}+\frac{e}{4M}\cdot M\le\frac{e}{8}+\frac{e}{4}\,.$$

It follows that

$$\begin{aligned}|T_{C,r}(f_i)-P(f_i)| &\le |T_{C,r}(f_i)-T_{C,r}(h_i)|+|T_{C,r}(h_i)-P(h_i)|+|P(h_i)-P(f_i)|\\ &\le\left(\frac{e}{8}+\frac{e}{4}\right)+\left(\frac{e}{8}+\frac{e}{4}\right)+\frac{e}{4}=e\,.\end{aligned}$$

6° If $l+1\le i\le n$, we have

$$\begin{aligned}T_{C,r}(f_i) &\ge T_{C,r}(h_i)\ge P(h_i)-\left(\frac{e}{8}+\frac{e}{4}\right)\ge\int_Z g_i-\frac{1}{2}\\ &=\int_Y g_i-\int_{Y-Z} g_i-\frac{1}{2}\ge P(g_i)-G\cdot m(Y-Z)-\frac{1}{2}\\ &\ge 2N-\frac{e}{8}-\frac{1}{2}\ge N\,.\end{aligned}$$

We have thus proved the Lemma. ■

3.3. THEOREM. *Let $(X, \boldsymbol{B}, m)$ be a non-atomic measure space. Then there exist a $*$-finite subset Γ of $^*X$ including X and a $*$-natural number $\rho\in{}^*\boldsymbol{N}$ such that*

$$\int_X f\,dm \doteqdot {}^*T_{\Gamma,\rho}({}^*f)=\frac{1}{\rho}\,{}^*\!\sum_{\xi\in\Gamma}{}^*f(\xi)\quad(\text{in } {}^*\overline{\boldsymbol{R}})$$

for all real valued measurable functions f on X having definite *integral value.*

Proof. Let E (resp. E_P) be the set of positive measurable (resp. integrable) functions on X and $\mathscr{S}(X)$ the set of finite subsets of X. Consider the binary formula $\phi(x, y)$:

$$x=(f, e, N, a)\in E\times \boldsymbol{R}^{++}\times \boldsymbol{N}\times X\wedge y=(C, r)\in \mathscr{S}(X)\times \boldsymbol{N}\wedge$$
$$a\in C\wedge[f\in E_P\rightarrow|P(f)-T_{C,r}(f)|\leq e]\wedge[f\in E-E_P\rightarrow T_{C,r}(f)\geq N]\,.$$

Lemma 3.2 shows that $\phi(x, y)$ is concurrent in $\mathscr{U}$, so there exist Γ and ρ with required properties for positive measurable functions. For a general measurable function f having definite integral value, it sufficies to write $f=f^+-f^-(f^+, f^-\in E, f^+\in E_P$ or $f^-\in E_P)$ and to apply the Theorem to f^+ and f^-. ■

3.4. Corollary. *Let $(X, \boldsymbol{B}, m)$ be a non-atomic quasi-measure space. Then there exist a $*$-finite subset Γ of $^*X$ including X and $\rho\in {}^*\boldsymbol{N}$ such that*

$$m(A)\doteqdot\frac{1}{\rho}\,{}^*|{}^*A\cap\Gamma|\quad(\text{in } {}^*\bar{\boldsymbol{R}}^+)$$

for all $A\in \boldsymbol{B}$, where $^*|\ |$ is the $*$-cardinality of a $*$-set.

Proof. It suffices to consider only simple functions $f_1, \cdots, f_n$ in Lemma. ■

3.5. Corollary (Henson [4]). *If m is a probability measure (resp. quasi-measure) in Theorem 3.3 (resp. Corollary 3.4), then we can take $\rho={}^*|\Gamma|$.*

Proof. Put $\gamma={}^*|\Gamma|$. We have

$$\frac{\gamma}{\rho}=\frac{1}{\rho}\,{}^*|{}^*X\cap\Gamma|\doteqdot m(X)=1$$

and

$${}^*T_{\Gamma,\gamma}({}^*f)=\frac{\rho}{\gamma}\,{}^*T_{\Gamma,\rho}({}^*f)\doteqdot\int f\,dm\quad(\text{in } {}^*\bar{\boldsymbol{R}}^+)$$

for positive measurable (resp. measurable simple) functions f. ■

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